Problem Sheet 4

Q1 - Classification

Linear — Only if dependant variable & derivatives are linear.

Autonomous — no dependence on t in the RHS
order too!

→ $\frac{dx}{dt} = \sin(t)x$ is ~~not~~ linear ✓ but non-autonomous ✓ 1st order ✓ as only first derivative.

→ $\dddot{x} = \frac{d^3x}{dt^3} = t\cos x$

Linearity — linear ✗ ↗ Trigonometric function makes it non-linear.
Autonomous — no as dependence on t ✓
Order — 3 as is third derivative ✓

→ $\ddot{x} = \frac{d^2x}{dt^2} = t$

Linearity — linear ✓
Autonomous — no as dependence on t ✓
Order — 2 as is second derivative ✓

→ $\frac{dy}{dx} = xy$ Instead of using As y is not to any power, there, we are using and not e.g., cos or sin, eqn x. x is dependant on RHS. is linear.

Linearity — non-linear as y also RHS ✗
Autonomous — yes ✗
Order — 1 ✓

Q1 - Into Systems of 1st order Autonomous

1 → $\frac{dx}{dt} = \sin(t)\,x$ (set $y = t$)

$\frac{dy}{dt} = 1$ ✓ $\frac{dx}{dt} = \sin(y)\,x$ ✓

↑

Add in (could also write as $\frac{d}{dt} y = 1$)

2 → $\dddot{x} = t + \cos(x)$ (rewrite)

$\frac{d^3x}{dt^3} = t + \cos(x)$

~~Introduce both $\dot{x} = t$ $\left(\frac{d^2x}{dt} = t\right)$ and~~

~~$z = t$ as equations. Then have~~

~~$\frac{dz}{dt} = 1$ (similar to previous solution)~~

~~$\frac{dx}{dt} = y$ (not sure why), then~~

~~$\frac{dy}{dt} = \frac{d^2x}{dt} = z$~~

⚠ Needs revision!

This is solution for third

Taken 1h to do Q1

PTO

4 → $\frac{df}{dx} = xf$ already first order. To make autonomous, need to remove dependence on x RHS.

Set $y = x$ (here, x in place of t)

$\frac{dy}{dx} = 1$ $\frac{df}{dx} = yf$

No solution on scan, can't check if right.

3 → $\ddot{x} = t = \frac{d^2x}{dt^2}$ 2nd order, also non-autonomous

If we introduce $y = \frac{dx}{dt}$, then we can change the $\ddot{x}$ which is currently 2nd order into $\dot{y}$, which is first order.

$y = \frac{dx}{dt}$ $\frac{d^2x}{dt^2} = \frac{dy}{dt} = \frac{d}{dt}\frac{dx}{dt}$

$= t$

Now have 2 first order, in terms of t so still non-autonomous. As have used y already, we introduce $z = t$, and $\frac{dz}{dt} = 1$, then replace all instances of t on RHS with z.

$y = \frac{dx}{dt}$ ✓ $\frac{dy}{dt} = z$ ✓ $\frac{dz}{dt} = 1$ ✓

Q2 Solution Techniques

As now reduced to first-order autonomous ODEs, some analytical methods exist.

Analytical Methods –

Direct Integration
Separable Functions
Linear Differential Equations

Direct integration is of form

$$\frac{dy}{dx} = g(x)$$

$$f(x) = \int_{x_0}^{x} g(x')\,dx'$$

Separable is of

$$\frac{dy}{dx} = f(x)\,g(y) \longrightarrow \frac{dy}{g(y)} = f(x)\,dx$$

$$\int \frac{dy}{gy} = \int f(x)\,dx$$

PTO

Linear difference inc. interesting factor μ

$$\frac{dy}{dx} + P(x)y = g(x)$$ → Linear as derivative 1st power, no sin, cos, etc.

$$\mu(x)\frac{dy}{dx} + \mu(x)p(x)y = g(x)\mu(x)$$

Integrating factor → $\mu(x)$

Choose $\mu(x)$ such that $\frac{d\mu}{dx} = \mu(x)p(x)$ (*), then

$$\mu(x)\frac{dy}{dx} + \frac{d\mu}{dx}y = g(x)\mu(x)$$

$$\frac{d}{dx}\left(\mu(x)y\right) = g(x)\mu(x)$$

$$y = \frac{1}{\mu(x)}\int^{x} g(x')\mu(x')\,dx'$$

From here, μ can be found from * as

$$\frac{d\mu}{dx} = \mu(x)p(x) \rightarrow \mu = \exp\int^{x} p(x')\,dx'$$

$y = \ldots$ so complicated!

$\mu(x) \times P(x) = \mu'(x)$ to get formula for $\mu(x)$

$\frac{\mu'(x)}{\mu(x)} = P(x)$ is integrated to get rid of derivative

$$\int \frac{\mu'(x)}{\mu(x)}\,dx = \int P(x)\,dx$$

$\ln(\mu(x)) = \int p(x)\,dx$ Don't care about constant here.

Then take exponent of both sides:

$$\exp(\ln(\mu(x))) = \exp\left(\int p(x)\,dx\right)$$

$$\mu(x) = \exp\left(\int p(x)\,dx\right)$$

This will give the integrating factor

1.5h

→ $\frac{dx}{dt} = rx$ separable

$$\frac{dx}{x} = r\,dt$$ ✓

$$\int_{x_0}^{x} \frac{dx}{x} = \int_{t_0}^{t} r\,dt$$

$$\int_{x_0}^{x} \frac{1}{x}\,dx = \int_{t_0}^{t} r\,dt$$

$$\Big[\ln(x)\Big]_{x_0}^{x} = \Big[rt\Big]_{t_0}^{t}$$

$$\ln(x) - \ln(x_0) = rt - rt_0$$

$$\ln(x/x_0) = r(t - t_0) \qquad t_0 = 0$$

$$\ln(x/x_0) = rt$$

$$x/x_0 = e^{rt}$$

$$x = x_0 e^{rt}$$

Q2 Cont'd

$\dot{x} = t \implies \frac{dx}{dt} = t$ direct

$x(t) = g(t)$

$$x(t) = \int_{x_0}^{x} g(t')\,dx'$$

$$= x_0 + \int_{t_0}^{t} t'\,dt'$$

$$= x_0 + \left[\tfrac{1}{2}t^2\right]_{t_0}^{t}$$

$$= x_0 + \tfrac{1}{2}t^2 - \tfrac{1}{2}t_0^2$$

$$= x_0 + \tfrac{1}{2}(t^2 - t_0^2) \quad t_0 = 0$$

$$= x_0 + \tfrac{1}{2}t^2$$

$\dot{x} = 1 + tx$ not separable, not direct so is eq linear difference.

$\frac{dx}{dt} = 1 + tx$ $f(x, t)$

$\frac{dx}{dt} - P(t)x = 1$

Call it t

Find integrating factor $\mu(t)$:

$$\mu \frac{dx}{dt} - \mu t x = \mu$$ drop (t) references

× Giving up for now.

TBC Q2cd, 3–5 ?

Seem like good revision questions