Problem Sheet 2 COMP6216

Q1 Taylor Expansions

Recall general form:

$$f_k(a) = \sum_{n=0}^{k} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Recall ~~ft~~ $\frac{dy}{dx}(\exp(x)) = \exp(x)$.

At $x=0$, $f_\infty(a) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x-0)^n$

$$= \frac{f^{(0)}(0)}{0!}(x-0)^0 + \frac{f^{(1)}(0)}{1!}(x-0)^1 + \dots$$

$$= \frac{f(0)}{1}(1) + \frac{f'(0)}{1}(x) + \frac{f''(0)}{2}x^2 + \dots$$

As $f(x) = f'(x) = f''(x), \dots$ for e^x, and $e^0 = 1$, substitute 1.

$$= \frac{1}{1}(1) + \frac{1}{1}x + \frac{1}{2}x^2 + \dots$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \dots$$

For $f(x) = \sin(x)$,

$f'(x) = \cos(x)$,

$f''(x) = -\sin(x)$,

$f^{(3)}(x) = -\cos(x)$,

$f^{(4)}(x) = \sin(x) = f(x)$ and thus loops.

$$f_{\infty}(a) = \sum_{n=0}^{\infty} \frac{f(n)}{n!}(x-a)^n$$

$$= \frac{f(0)}{0!}x^0 + \frac{f'(0)}{1!}(x-0)^1 + \frac{f''(0)}{2!}(x-0)^2 + \dots$$

$$= \sin(0) + \cos(0)x - \frac{\sin(0)x^2}{2!} - \frac{\cos(0)x^3}{3!} + \dots$$

$$= 0 + x - \frac{0x^2}{2!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$

As $\sin(0)$ always $= 0$, and derivatives are $0, 1, -0, -1, 0, 1, \dots$

$$f(x) = \frac{1}{5+x}$$

Quotient rule? $\times$ seems a bit long.

Exponent rule? $\checkmark$ Not too useful either...

$$\frac{1}{5+x} = (5+x)^{-1}$$

Reciprocal rule:

$$\left[\frac{1}{u(x)}\right]' = \frac{u'(x)}{u(x)^2}$$

$$f'(x) = \frac{\frac{dy}{dx}(5+x)}{(5+x)^2} = -\frac{1}{(x+5)^2}$$

$$f''(x) = \frac{\frac{dy}{dx}(x+5)^2}{((x+5)^2)^2}$$

$$= \frac{\frac{dy}{dx}(x^2+10x+25)}{(x+5)^4}$$

$$= \frac{2x+10}{(x+5)^4}$$

Need to use the power rule?

Power rule

$$[u(x)^n]' = n \times u(x)^{n-1} \times u'(x)$$

$$f''(x) = -\frac{d}{dx}\left[\frac{1}{(x+5)^2}\right] = -\frac{d}{dx}\left((x+5)^{-2}\right)$$

$$= -(-2)(x+5)^{-3}(1)$$

$$= \frac{2}{(x+5)^3}$$

$$f'''(x) = \frac{d}{dx}\left[\frac{2}{(x+5)^3}\right]$$

$$= 2\frac{d}{dx}\left[(x+5)^{-3}\right]$$

$$= 2(-3)(x+5)^{-4}(1)$$

$$= -6(x+5)^{-4}$$

$$= \frac{-6}{(x+5)^4}$$

Now to start Taylor!

$$f_3(0) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!}$$

$$= 1/5 - 1/25 + 2/125 - 6/\cancel{250}\,625$$

$$= 104/625 = 0.1664.$$

Misread the question

$$f_3(x) = \frac{x}{5+x} - \frac{5}{(x+5)^2} \quad \times$$

Apparently, should recognise this is a geometric series.

Q2 Multivariate Taylor

$$f(x,y) = x^2 + y^2 \qquad \frac{df}{dx} = 2x + y^2 \qquad \frac{df}{dy} = x^2 + 2y$$

$$H = \begin{bmatrix} \frac{df}{dx^2} & \frac{df}{dxdy} \\ \frac{df}{dydx} & \frac{df}{dy^2} \end{bmatrix}$$

$$= \begin{bmatrix} 2 + y^2 & 2x + 2y \\ 2x + 2y & x^2 + 2 \end{bmatrix}$$

$$f_2(x,y) = f(0,0) \qquad \leftarrow \text{No vars}$$

$$+ \frac{df}{dx}(0,0)x + \frac{df}{dy}(0,0)y \leftarrow 1 \text{ var}$$

$$+ \frac{\frac{df}{dx^2}(0,0)x^2}{2!} + \frac{\frac{df}{dy^2}(0,0)y^2}{2!} + \frac{\frac{df}{dxdy}(0,0)xy}{2!}$$

$$+ \frac{\frac{df}{dydx}(0,0)yx}{2} \leftarrow 2 \text{ vars}$$

$$= 0$$

$$+ 0x + 0y$$

Eval partial derivative at (0,0), using Hessian.

$$+ \frac{(2)x^2}{2!} + \frac{(2)y^2}{2!} + 0xy + 0yx$$

$$= \frac{2x^2}{2} + \frac{2y^2}{2}$$

$$g(x,y) = \exp(x + y) \qquad \frac{dg}{dx} = x e^x e^y$$

$$= e^x e^y \qquad \frac{dg}{dy} = e^x y e^y$$

$$H = \begin{bmatrix} \frac{df}{dx^2} & \frac{df}{dy\,dx} \\ \frac{df}{dx\,dy} & \frac{df}{dy^2} \end{bmatrix}$$ ← Wrong? Need to swap so premultiply by var?

$$= \begin{bmatrix} x^2 e^x e^y & x e^x y e^y \\ x e^x y e^y & y^2 e^x e^y \end{bmatrix}$$

$$f_2(x,y) = f(0,0)$$

$$+ \frac{df}{dx}(0,0)x + \frac{df}{dy}(0,0)y$$

$$+ \frac{\frac{df}{dx^2}(0,0)\,x^2}{2!} + \frac{\frac{df}{dy^2}(0,0)\,y^2}{2!} + \frac{\frac{df}{dx\,dy}(0,0)\,xy}{2!}$$

$$+ \frac{df}{dy\,dx}(0,0)\,yx$$

$$= 1 + 0x + 0y + \frac{0x^2}{2} + \frac{0y^2}{2} + \frac{0xy}{2} + \frac{0yx}{2}$$

$$= 1 \quad ?$$

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Q3 Numerical Differentiation

Est. df/dx of $f(x) = \exp(x)$ for $x = 0$

Taylor expansion at $f(x-h), f(x+h)$, compare $\bar{c}\, f(x)$

Forward is $f'(x) \approx \frac{1}{h}(f(x+h) - f(x))$

Central $\frac{1}{2h}(f(x+h) - f(x-h))$

Backward $\frac{1}{h}(f(x) - f(x-h))$

$h = 0.1$

Forward: $\frac{1}{0.1}(\exp(0+0.1) - \exp(0))$

$10(0.1052)$ (4.s.f.)

$= 1.052$

Central: $\frac{1}{2(0.1)}(\exp(0+0.1) - \exp(0-0.1))$

$20((0.1052) - (0.9048))$

~~10(1.0~~

$5(0.2003335)$

1.0016675

Back: $\frac{1}{0.1}(\exp(0) - f(0-0.1))$

0.9516

Can now take actual $\frac{df}{dx}(\exp(x)) = \exp(x)$

$\exp(0) = 1$.

Forward is	+0.052
Central	+0.002
Back	−0.048

Central is much more accurate.

<u>Q4 Higher Order Finite Difference</u>

Central difference accurate to second order. To yield higher orders, introduce $f(x+2h)$ and $f(x-2h)$, to e.g. 3rd order

$$f_3(x+2h) = f(x) + 2hf'(x) + h^2 f''(x) + \frac{1}{3}h^3 f^{(3)}(x) + \frac{16}{4!}h^4 f^{(4)}(x)$$

$$f_3(x-2h) = f(x) - 2hf'(x) + h^2 f''(x) - \frac{1}{3}h^3 f^{(3)}(x) + \frac{16}{4!}h^4 f^{(4)}(x)$$

Using this, can calculate:

$$f(x+2h) - f(x-2h)$$

$$f(x) - f(x) + 2hf'(x) + 2hf'(x) + h^2 f''(x) - h^2 f''(x) + \frac{1}{3}h^3 f^{(3)}(x) + \frac{1}{3}h^3 f^{(3)}(x)$$

Wrong! $\frac{2^3}{6} + \frac{2^3}{6} = \frac{16}{6}$

$$= 4hf'(x) + \frac{16}{6}h^3 f^{(3)}(x) + (\text{no } h^4) + O(h^5) \quad (2)$$

$f(x+h) - f(x-h)$ (from lecture slides)

$$= 2hf'(x) + \frac{1}{3}h^3 f^{(3)}(x) + (\text{no } h^4) + O(h^5) \quad (1)$$

Then combine to eliminate the h^3 term.

$16/6 = 8/3$ but also have $f'(x)$ term.

Get both (1) & (2) in terms of f'

$$f(x+h) - f(x-h) = 2hf'(x) + \frac{1}{3}h^3 f^{(3)}(x) + O(h^5)$$

$$\frac{f(x+h) - f(x-h)}{2h} = f'(x) + \frac{1}{6}h^2 f^{(3)}(x) + O(h^4)$$ As div by h

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} - \frac{1}{6}h^2 f^{(3)}(x) - O(h^4) \quad (1')$$

$$f(x+2h) - f(x-2h) = 4hf'(x) + \frac{16}{6}h^3 f^{(3)}(x) + O(h^5)$$

$$\frac{f(x+2h) - f(x-2h)}{4h} = f'(x) + \frac{2}{3}h^2 f^{(3)}(x) + O(h^4)$$

$$f'(x) = \frac{f(x+2h) - f(x-2h)}{4h} - \frac{2}{3}h^2 f^{(3)}(x) - O(h^4) \quad (2')$$

Can eliminate h^2 term $\bar{c}$ subtraction of $4(1')$ from $(2')$

$(2') - 4(1')$

$$-3f'(x) = \frac{f(x+2h) - f(x-2h)}{4h} - \frac{2}{3}h^2 f^{(3)}(x) - O(h^4)$$
$$- \left(\frac{2f(x+h) - 2f(x-h)}{h} - \frac{2}{3}h^2 f^{(3)}(x) - O(h^4) \right)$$
$$= \frac{1}{4h}\left(f(x+2h) - f(x-2h) - 8f(x+h) + 8f(x-h) \right) + O(h^4)$$

Divide by -3 and should be there?

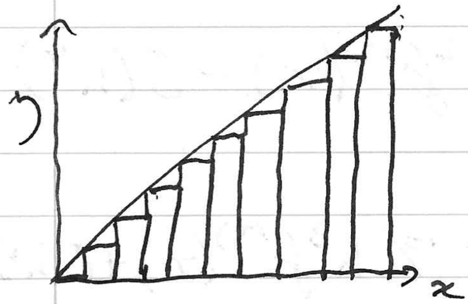
Q5 Numerical Integration

$\int_0^1 f(x)\,dx$ for $f(x) = x$

$= \left[\frac{1}{2} x^2 \right]_0^1$

$= \frac{1}{2}$

Square Method:



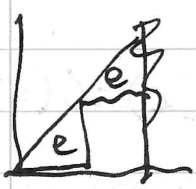
$\sum_{i=0}^{n-1} f_i (x_{i+1} - x_i)$

In interval $[0,1]$, using square method, there will be $1/h$ intervals total. Area of a triangle is $\frac{1}{2}$ base $\times$ height.

If $h = 1$, error would be $\int_0^1 x\,dx = \left[\frac{1}{2}x^2\right]_0^1$

$= 1/2.$

For $h = 0.5$, error would be $\int_0^{0.5} x\,dx + (\int_{0.5}^1 x\,dx - 0.5\times0.5$



$\frac{1}{2}(0.5^2) + ((\frac{1}{2} - \frac{1}{2}(0.5)^2) - 0.5^2$
$= 1/4.$

For $h = 0.25$ $\int_0^{0.25} x\,dx + (\int_{0.25}^{0.5} x\,dx - 0.25\times0.25)$
$+ (\int_{0.5}^{0.75} x\,dx - 0.5\times0.5)$
$+ (\int_{0.75}^{1} x\,dx - 0.75\times0.75)$

$$\text{error} = \sum_0^n \left(\int_{nh}^{nh+h} f(nh)\,dx - f(n)h \right)$$

where n is num intervals $n = 1/h$ for $[0;1]$ or

$n = (b-a)/h$ otherwise.

As $h \to 0$, error also $\to 0$.

If modelling = trapeziums, we can perfectly approximate $f(x)=x$ as it is a linear function.

Square integration – I think this refers to drawing a square under the ~~...~~ line. In this instance, only functions y=constant would be exact. ~~As $h \to 0$, squares would be valid~~

<u>Unsure about above.</u>

Trapezoid rule would approximate anything linear, e.g. $y=x$.